Linear Air Trough

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The construction and operation of a linear air trough, a device for floating small rectangular blocks (called gliders) on an air film, is described. The apparatus is useful for demonstrations and laboratory study of one-dimensional particle mechanics. The very small friction that is present is due to the viscosity of the air film, and causes the speed of a glider to decay with a time constant of the order of a few hundred seconds. A type of bumper has been designed which yields a coefficient of restitution greater than 0.99. Ten of these air troughs have been used in a student laboratory for a year, and have proved quite successful in experiments involving Newton's laws, collisions, damped harmonic motion, and motion on an incline.

I. INTRODUCTION

PUCKS that float on a film of gas have been described in the literature,[1] and have been widely used to demonstrate and study motion in two dimensions. For qualitative measurements, the gas-supported puck is a very suitable and, indeed, a very striking device. Two methods for admitting the gas to the base of the puck have been used. One of these uses a small tube from an overhead supply of gas. The other uses a supply of gas carried along by the puck itself. In the latter case, the gas supply may consist of a compressed gas like a small carbon dioxide capsule, or a supply of solid carbon dioxide may be carried in a container and the gradual subliming of the solid furnishes the required gas supply. Rubber balloons have also been used.

There are obvious limitations imposed by the use of all of the above systems of supplying the necessary gas. A much better arrangement would be to inject the gas through numerous holes in the plate on which the pucks are to float. However, this system would become involved, from a mechanical point of view, especially if the flat surface were very large.

Injecting the gas through holes in the surface on which the puck glides does become practical for a linear device, and such a device is herein described. We shall adopt the name *air trough* for such a piece of apparatus, and shall call the moving blocks *gliders*. While it is true that some features of collisions of objects are lost if only the one-dimensional case is studied, many phenomena may be studied with precision and ease with the linear system. It seems unlikely that, in an introductory laboratory, very much time would need to be spent on collisions in two dimensions if the characteristics of collisions in one dimension had been studied quantitatively with an air trough.

[1] Robert G. Marcley, Am. J. Phys. **28**, 670 (1960).

II. DESCRIPTION OF THE APPARATUS

The trough is conveniently made with plane sides with a 90° angle between them. To supply the gas (e.g., air) at regular intervals along the trough, two rows of equally spaced inserts with small holes are placed along each side. These tap the air supply in the two manifolds that run the full length of the trough.

While the original models were fabricated by welding together commercially available "U" and right-angle aluminum extrusions, later models were made with a special aluminum extrusion. To maintain the necessary straightness and rigidity, the extrusion is fastened to an aluminum I-beam by means of screws and epoxy resin. For an 8-ft-long trough, a 4-in. I-beam provides sufficient rigidity, while a 12-ft trough requires a 6-in. I-beam. A cross section of the extrusion and I-beam is shown in Fig. 1.

Also shown in Fig. 1 is a cross section through four of the numerous inserts which feed air from the common manifold to the surface on which the glider moves. These inserts are 0.063-in.-o.d. stainless steel tubing and have a 0.006-in.-diam coaxial hole. Such tubing is commercially available. These inserts are spaced along the trough at intervals of one inch. They are staggered in the four rows so that, with the proper length of

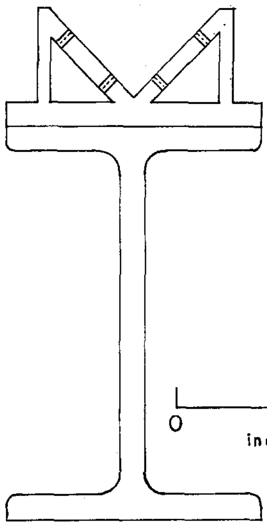

FIG. 1. A cross section of the special aluminum extrusion and the I–beam to which it is fastened. Also shown are cross sections through the stainless-steel inserts with their small holes through which the air flows from the two manifolds.

glider, only one jet of air at a time is uncovered as the glider moves along the trough. This is desirable because of a small Bernoulli action on the end surface of the glider.

Because of the rather large number of inserts that are required, the following method was used to make them in quantity:

(1) A suitable length of steel tubing, e.g., $\frac{5}{8}$ in. i.d., $\frac{3}{4}$ in. o.d., is plugged at one end and partly filled with zinc. While the zinc is molten, suitable lengths of the stainless steel tubing are forced down into the liquid. If the whole is kept hot, the molten zinc will rise up around the small tubing. Some 70 pieces of the stainless steel tubing may be so inserted into a $\frac{5}{8}$-in.-i.d. steel tube.

(It has been reported to the authors by Professor Malcolm Correll of the University of Colorado that epoxy resin may be substituted for zinc for holding the inserts in place. The epoxy may be removed with acetone. This procedure avoids the possibility of the inserts being attacked by the acid when the zinc is removed.)

(2) The steel tubing, filled with zinc and small tubing, is next cut into wafers about $\frac{3}{16}$ in. long.

(3) A surface grinder or abrasive wheel is then used to smooth the wafers. A grinding wheel with loose grit should not be used as the small holes are apt to become closed. Most of the 0.006-in. holes, at this stage, should be open at both ends.

(4) To clear out the burrs from the remaining holes a 0.005-in.-diam tungsten wire may be used, or, if a miniature sand blast is available, this may be effectively employed.

(5) The zinc is now etched out of the wafers to free the stainless-steel tubing, using hydrochloric acid. Care should be taken to be sure that the inserts are not also attacked by the acid.

(6) To round the edges on the inserts and thus make it easier to force them into holes in the aluminum, they may be tumbled. One may place a large number in a jar lined with carborundum paper and rotate this slowly about a horizontal axis. Tumbling for a period of 12 to 24 h at 1 or 2 rps should be sufficient.

Before the inserts are put into the holes, the trough should be well fastened to a suitable I-beam, after which the sides of the trough are planed. It is found that the gliders, when floating on an air film and nearly stationary, are sensitive to slopes of 1 part in 20 000. Thus, the requirements on the variations of the planed surface are $\pm$ a few thousandths of an inch in the length of the trough.

The holes for the inserts are conveniently drilled with a jig either before or after the trough is planed. These holes should be slightly smaller than the inserts so that the latter will be held firmly by the aluminum.

The manifolds in the extrusion are closed with suitable end plates. Air is admitted to these manifolds by connections to an air line furnishing a pressure of 5 to 40 psi. The air should be filtered to avoid stoppage of the small holes.

It is convenient for many experiments to have stops with suitable bumpers for the gliders at each end of the trough. If the I-beam is made several inches longer than the trough, space will be available at each end to fasten an aluminum block in which a spring bumper, shortly to be described, may be mounted.

It is desirable to have a means of changing the inclination of the trough by a definite amount. This may be done by equipping one end of the trough with a screw on which is mounted a dial with a scale. A stationary scale indicates the integral number of turns. Near the other end of the trough a cross piece is fastened which in turn can rotate in two metal blocks that may be rigidly clamped to a bench or table. A suitable choice of pitch of screw and length of lever arm gives a change of slope of 1 in 1000 for one rotation of the screw.

III. AUXILIARY EQUIPMENT

(1) The gliders may be made of materials such as aluminum, brass, and steel. The two faces

that fit in the trough should be ground straight and have the proper angle with respect to each other. The top corner of the gliders is best truncated by milling a flat surface. This not only serves the purpose of easily identifying the proper orientation of the glider but, also, serves as a convenient surface on which to fasten various auxiliary devices. It is desirable also to provide a means of lifting the heavier gliders to make it easier to place them in the trough without causing damage. A bail type of handle is easily mounted at each end.

Suitable bumpers on the ends of the gliders as well as at the ends of the trough are very desirable. A satisfactory design is shown in Fig. 2. The spring material used in the bumper is available commercially as clock spring stock. Plug C (Fig. 2) has two flat places to take the double leaf springs D, when C is forced into B. The two springs thus take on the shape of a "U" with the thinner material (0.006×0.375 in.) being backed up by the thicker spring (0.016×0.437 in.) which in turn is backed up by a screw. The spacings should be such that the thin spring can distort by about 2 mm before touching the thick spring, and this can distort about $1\frac{1}{2}$ mm before bottoming on the screw. This combination avoids distorting the springs beyond their elastic limit at hard impacts.

The energy loss at the ends is very much less for the thin spring material than when the thicker spring is used. With the steel glider (2.1 kg), and using 0.016-in.-thick springs on the glider and stops at the ends of the trough, the loss of energy on impact with a velocity of about 30 cm sec^{-1} was about 12%. Changing all bumpers to the 0.006-in.-spring material reduced the loss at the ends to less than 1%.

(2) For studying the motion of a glider, it is often desirable to have a spark record. A surface for holding the paper, that runs the full length of

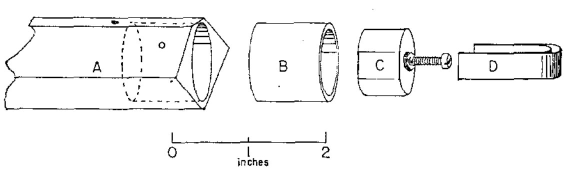

FIG. 2. Design of the bumper used in each end of most of the gliders as well as in the stops at the ends of the trough. Using these bumpers, collisions are more than 99% elastic.

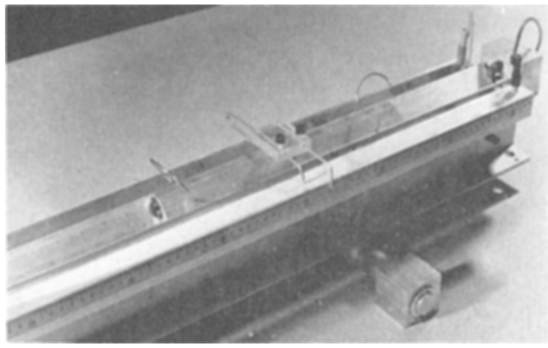

FIG. 3. The arrangement for using the insulated rails for making a spark record on waxed paper, is here shown. Also attached to the gliders is a pointer for visual determination of the position of the glider. Note also the double-leaf spring used as a bumper.

the trough, as well as an electrically insulated rail are easily mounted on the base of the extrusion as shown in Fig. 3. An insulated metal probe, mounted on the glider, carries the high potential from the spark rail to its pointed end which moves along just above the waxed paper.

A convenient rate of sparking is 1 per half-second. This may be achieved with a cam, operating a microswitch, and driven by a small synchronous motor. An automobile high tension coil is a convenient source of high voltage.

(3) On the side of the angle piece on which the waxed paper is mounted, a suitable metric scale may be fastened. A pointer mounted on the glider provides a means of determining its position: (See Fig. 3).

(4) For studying damped harmonic motion, long helical springs may be used to fasten to each end of a glider, the other end of the springs being fastened to the ends of the trough. With a period of about 5 sec, the logarithmic decrement is about 0.02 for a steel glider. These springs may be wound on a lathe from piano wire.

(5) To study forced, damped simple harmonic motion, a variable speed, geared motor may be used. An amplitude of about 1 cm is convenient. A crank on the motor is attached to one of the gliders. Wires from this glider connect to a similar glider near the other end of the trough. This second glider is fastened by means of a spring to the end of the trough. Long helical springs then connect from these two driven gliders to the third glider at the center of the trough. Thus the end gliders are driven with a definite amplitude

at a variable frequency. The motion of the driven glider may then be studied.

(6) The driven glider, in 5 above, may be provided with a damping proportional to its velocity by mounting on a glider, made from a piece of aluminum angle, a horse-shoe shaped permanent magnet. It is desirable to place soft iron inserts into the aluminum to produce a strong magnetic field that links the aluminum of the extrusion. Means may also be provided for raising and lowering the magnet on the glider, thus changing the damping. By such means, the Q of the system, with suitable springs, may be changed from about 2 to 120.

(7) To study action and reaction, gliders may be provided with a mechanism to blow them apart with powder caps used in some toys. One may obtain such caps with sticky adhesive on the back at toy stores. Phenolic insulated pieces, in the form of piston and cylinder, are substituted for the bumpers in the gliders. An O-ring, mounted in the piston provides a convenient seal for the firing chamber. The piston contains a pointed metal rod with a suitable connection so that a high potential may be applied externally which will pass through and thus ignite the powder of the cap. The cap is stuck over the end of the pointed rod which in turn is flush with the end of the phenolic. The cylinder contains a metal screw in its base to which the spark jumps after passing through the cap. Suitable adjustments may be made so that after the cap is exploded and the gliders start off in opposite directions and return after bumping off the ends of the trough, they stick to each other when they come together.

(8) To study random motion of particles, a variable speed, sinusoidal drive may be provided by a crank-and-flywheel driven piston mounted at one end of the trough. A number of short gliders are placed in an inclined trough and they are kept in motion by this source of "heat" at the lower end. The frequency of drive should be a few strokes a second, and a large moment of inertia should be provided to minimize loss due to collision with the gliders. This piece of auxiliary apparatus could be combined with that in (5) above to provide a source for forced simple harmonic motion.

(9) To study normal modes and other coupled systems, it is desirable to have a spring that may be extended some distance and yet behave properly when compressed. Such a spring may be made from the same clock spring material as is used to make the bumpers. A section of the 0.006×0.375-in. leaf spring, cut about 24-in. long, bent into the shape of a semicircle gives a period of a few seconds when attached to the 3-in. long aluminum gliders. To attach this spring to the gliders, holes are drilled, say, a quarter inch from each end, then the last half inch of the spring is bent at right angles. The spring is then screwed down to the tops of the gliders. Two equal gliders so attached may be made to proceed along the trough in a "measuring worm" kind of motion.

(10) To study damped motion as such, the usual glider is not satisfactory because of the low damping. While the glider described in 6 above may be used, it may be desirable, especially for demonstration purposes, to embed a magnet in a glider. This may be done by using square aluminum tubing, $1\frac{1}{4}$ in. on the outside, with a $\frac{1}{8}$-in. wall, and placing a bar magnet inside, say $\frac{3}{4}$-in. in diameter and 8 in. long. On a slightly tilted trough, such a glider soon reaches its terminal velocity.

(11) Accelerated motion may be studied when the trough is level by pulling a glider, say the 2.1-kg mass, with a calibrated spring. A spring of this sort is described by one of us (H. V. N.)[2] as used with the Maxwell top. Such a device may be fastened to the top of the 12-in. steel glider. The student may either pull the spring to a certain point, or it may be pulled by a Hero type engine working with air pressure.

IV. USE AND CARE OF THE APPARATUS

Experience in using 10 of these troughs, almost every day for a period of six weeks, has shown:

(1) With common air filters in the building compressed air line, no trouble was found in the small holes becoming stopped.

(2) Because of the small clearances (approximately 0.003 in. between trough and glider) it is necessary to have the surfaces clean. It was found sufficient to clean both gliders and trough at the

[2] H. V. Neher, Am. J. Phys. 30, 503 (1962).

beginning of a laboratory period with paper tissue and xylene.

(3) If the smooth surface of a glider is damaged, it may be readily ground flat again by rubbing it on fine carborundum paper backed by a flat surface. (We have had our aluminum gliders anodized, and those made of steel, plated with chromium. This treatment not only preserves the surface, but probably results in a surface less likely to be damaged by scratching.)

V. SOME GENERAL PROPERTIES OF THE AIR TROUGH

(1) The frictional forces retarding the motion of the gliders in the trough seem to be entirely due to viscous damping of the thin air film between the glider and the trough surfaces. As such, the frictional drag goes to zero at zero velocity. Further, the forces required to give the gliders a small velocity are extremely minute. As a consequence, the light aluminum gliders wander slightly due to disturbing influences, such as the Bernoulli forces arising from the jets of air blowing by the end faces. For this reason, brass or steel are more satisfactory materials from which to make most of the gliders. As is shown later, however, it is best in all cases to use reasonable velocities in experiments—up to 30 cm sec^{-1} for steel gliders, and up to 60 cm sec^{-1} for those made of aluminum.

(2) The amount by which the gliders are lifted by the air film, as a function of air pressure, may be determined by means of a dial indicator. In the case of the steel glider the results in Table I were found.

The figures in the last column were calculated from the data in the second column. The mass of the steel glider was 2.131 kg, and it was supported by 47 jets.

(3) With the help of the above data, the damping of the glider due to viscosity may be calculated. For a damping proportional to the velocity, the equation of motion, when the trough is horizontal, is

$$m\ddot{x}+(\eta A/d)\dot{x}=0, \qquad (1)$$

where η is the coefficient of viscosity of the air, A the area of the surfaces, and d the spacing. The solution of Eq. (1) in terms of x and t is

$$v=v_0e^{-t/\tau},$$

TABLE I.

Pressure	Vertical distance	Surface clearance
5 psi	0.0010 cm	0.0007 cm
10	0.0025	0.0018
20	0.0048	0.0034
30	0.0066	0.0047
40	0.0079	0.0056

where $\tau=md/\eta A$ is the characteristic time of the motion, or the time for the velocity to drop to $1/e$ of its initial value.

Data were taken with waxed paper and sparker for the steel glider going in both directions, with an air pressure of 23 psi. Using the data in Table I, the characteristic time was calculated to be 233 sec. The measured τ from the tape, averaging the two directions, was found to be 275 sec.

The characteristic time for the 3-in. aluminum glider is about 60 sec.

(4) Equation (1) may also be solved for v in terms of distance traveled. Thus,

$$mv(dv/dx)+(\eta A/d)v=0,$$

or

$$v=v_0-1/\tau x. \qquad (2)$$

This predicts that the velocity should decrease linearly with the distance traveled.

To test this, and also to find the losses due to the collision of gliders with the stationary ends, a second glider was clamped rigidly in the trough to vary the distance traveled between bounces. The velocities given in Fig. 4 are those found from stopwatch measurements between collisions, or for a number of collisions where the distance traveled between bounces was small.

It is seen that the velocity does really decrease linearly with distance traveled, even when many bounces or collisions occur. Thus in curve H where over 300 collisions took place, the velocity changed linearly with the distance. The behavior of curve A, taken with H indicates not only that the change of velocity is due to a damping proportional to the velocity, but the change of velocity is also proportional to the number of bounces. Thus the change of velocity at a bounce is independent of the velocity, and has a constant value. This change in velocity per bounce is found to be approximately -0.04 cm sec^{-1}. It is thus possible to construct a new curve (A′) which

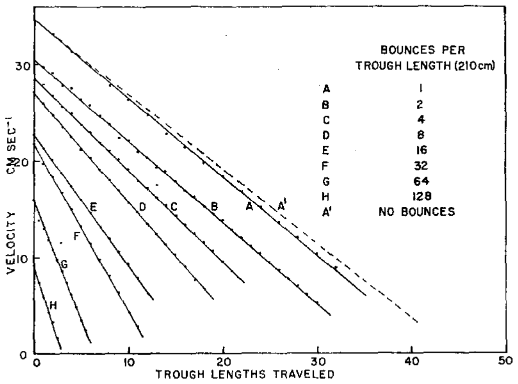

FIG. 4. Shown above is the velocity of the glider vs distance traveled for different numbers of collisions of the bumpers per unit distance traveled. The change of velocity with distance, due to the viscosity of the air film, is constant. Also, the change of velocity per bounce is constant, i.e., independent of velocity. The dotted line, A′, gives the expected velocity of the glider vs distance traveled in a very long, horizontal trough.

would represent the velocity vs distance for no bounces.

This curve A′ has a slope of 0.0036 $\sec^{-1}$ or $\tau=280$ sec. This value should be compared with the value of $\tau=275$ sec found from measurements made on a wax paper, using one trough length.

(5) As pointed out above, the change of velocity per bounce is independent of the velocity and has a constant value of about -0.04 cm $\sec^{-1}$ bounce^{-1}. Now the relative change of kinetic energy is $\Delta E/E=2\Delta v/v$. Hence $\Delta E/E$ per bounce is not constant, but has a value of about -0.31% per bounce near the beginning of curve A and about -1.6% per bounce at the lower velocities of about 5 cm $\sec^{-1}$.

Thus, where experiments are being performed in which losses due to damping or collisions are to be minimized, the larger velocities should be used. For example, at 30 cm $\sec^{-1}$, it requires a glider about 7 seconds to travel the length of the trough. Yet, the change of velocity in going this distance is only 0.83 cm $\sec^{-1}$, and when the glider collides with the end, it changes its velocity only 0.04 cm $\sec^{-1}$. On the other hand, if a velocity of 5 cm $\sec^{-1}$ were used on the full length trough, the change of velocity would be the same due to damping but would amount to a decrease of 16 to 17% of the initial velocity. The velocity change due to the bounce at the end would, however, still be less than 1%.

VI. SOME EXPERIMENTS

The number of experiments that may be performed with the device, herein described is limited only by the ingenuity of those who have worked with such a piece of apparatus. It has its fascinations for faculty as well as students.

Some of the experiments that have been found instructive are as follows:

1. One may equip the gliders with various means of repelling each other. The use of leaf spring bumpers has already been described. One may also use horseshoe magnets mounted in the ends. A bar magnet should not be used because of the eddy currents induced in the aluminum extrusions. One may also use the expansion of gas as described earlier in connection with the exploding gun powder in toy caps. Thus one may establish experimentally that, with equal gliders, the acquired velocities are always equal and opposite, and independent of the kind of force involved.

2. Inertial Mass vs *Gravitational Mass.* Let one mass m_2 strike a stationary mass $m_1 (m_1>m_2)$ somewhere along the trough, which is horizontal. There is one location of m_1 such that, m_2 and m_1, after collision, strike the ends at the same time. If the collisions are all nearly elastic, the returning masses will again collide close to the location of the original collision. Furthermore, m_2 will stop while m_1 will rebound with nearly the same speed it had originally. The process will repeat itself a number of times, if the system is not disturbed.

Assuming no loss of energy, it may be shown that the ratio of masses in the above experiments is

$$m_1/m_2=(2l_2/l_1)+1, \tag{3}$$

where l_1 and l_2 are the distances traveled by m_1 and m_2, respectively, before colliding with the ends. Using steel and brass gliders whose respective gravitational mass ratio was 3.905, the inertial mass ratio found from Eq. (3) was 3.84. This difference is in the direction to be expected from the losses along the track suffered by the two masses.

3. The acceleration of gravity. If the trough is tilted at a small angle θ to the horizontal, there are two obvious methods for determining the acceleration due to gravity. One, is to time the glider in going down the inclined plane, starting from rest. The other, is to allow the glider to rebound at the end bumper and measure the time for the glider to go back up the trough, reverse its direction, and strike the bumper a second time. This latter method is to be preferred since, to a first order of approximation, the effect of losses on the trough cancel out. Thus the time required to go to its turning point is less than it would be without losses. On the other hand, the time to return to the bumper is greater than it would have been without losses.

The following data were found with a tilt of the trough of $\theta=0.00200$, using a steel glider:

Distance of rebound	Time up and down	Calculated g
1.950 m	19.82 sec	9.93 msec^{-2}
1.495	17.43	9.84
1.004	14.30	9.83
0.790	12.75	9.72
0.474	9.88	9.71

4. *Further measurements to determine the characteristic time.* With an inclined trough, one may observe the successive points of reversal of the motion of the glider as it continues to rebound from the lower bumper. Were it not for losses, these points would, of course, all be identical. Hence the differences must be a measure of the losses. By solving the appropriate differential equation one finds that this Δx, to a first order of approximation, may be expressed as

$$\Delta x=[8/3(2g\theta)^{\frac{1}{2}}](x^{\frac{3}{2}}/\tau), \quad (4)$$

where x is the distance the glider travels up the trough, τ the characteristic time, while g and θ have their usual meaning.

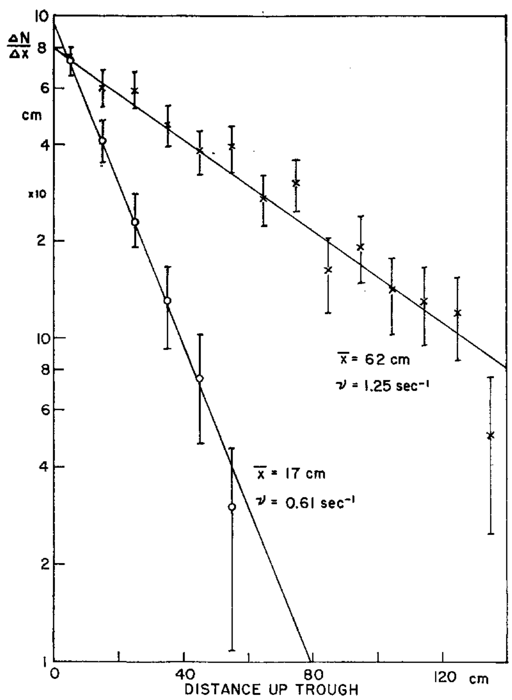

FIG. 6. On an inclined trough, a motor driven bumper, having a repetitious motion in a longitudinal direction at the lower end would be expected to give a random motion to the glider as it is repeatedly struck. The data in this figure were taken by dividing the length into uniform intervals and counting the number of times the glider reversed its motion in each interval. Randomness is indicated by the fact that the semilog plot gives a straight line.

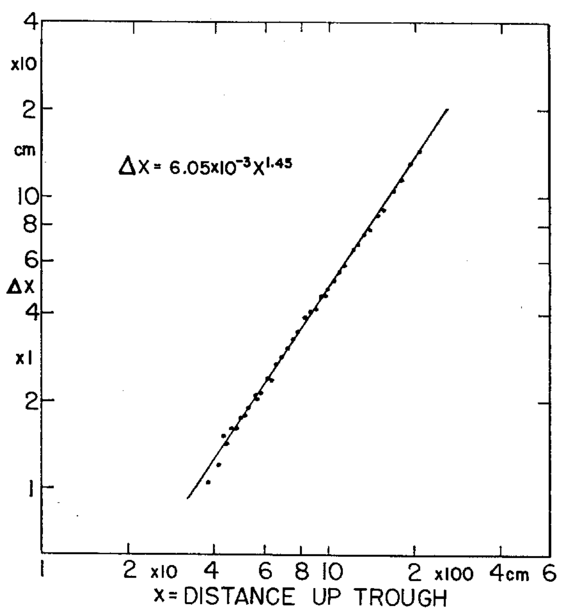

FIG. 5. For repeated bouncing-off the end bumper, with an inclined trough, the distance of reversal of motion of the glider decreases with each bounce. This constitutes another method of finding the characteristic decay time of the glider due to the air film.

The experimental values of Δx vs x are plotted in Fig. 5 in log–log coordinates. Here the tilt was $\theta=0.00200$, the glider was of steel, and an air pressure of 18 psi was used. The equation of the curve is

$$\Delta x=6.05\times10^{-3}\times1.45. \quad (5)$$

Taking the two constants of (4) and (5) to be equal, the value found for the characteristic time is

$$\tau=224 \text{ sec.}$$

This should be compared with the two values of 275 and 280 sec found previously. The difference may be accounted for from the difference in the air pressure used. The larger previous values of characteristic time were obtained with an air pressure of 23 psi. Using the data in Table I, the air film for this case was 3.84×10^{-3} cm thick. For the 18 psi used in the present experiment, the air film was 3.15 cm thick. Assuming a damping inversely proportional to the air film thickness, the value of 224 sec reduced to the same thickness of air, would become 274 sec. A high precision in the determination of this characteristic time should not be expected as differences in gliders, air pressure, foreign matter, etc., can all change the experimental value slightly.

5. If the variable-speed-motor-driven end-bumper described in Sec. III (8), is available, it is instructive to examine the energy distribution law for gliders which are in statistical equilibrium with this "heat source." A single glider is placed in the trough and the trough is tilted steeply enough to prevent the glider from reaching the high end, except rarely. Then, with the drive motor going at some convenient speed, the "heights" of successive bounces of the glider are measured, and tallied in, say, 10 cm intervals. A semi-log plot of this distribution should exhibit a straight line relation. Such distributions for two motor speeds, in approximately a 2:1 ratio, are shown in Fig. 6.

ACKNOWLEDGMENTS

We would like to take this opportunity to express our appreciation to the many individuals who have helped, in one way or another, to carry through this project; in particular, the personnel in the machine shop for their skillful work. We also wish to acknowledge the financial assistance of the Ford Foundation in supporting the development of such instructional laboratory equipment.

Acceptable Solutions and Boundary Conditions for the Schrödinger and Dirac Equations*

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It is shown that the condition of finiteness at the origin, usually imposed on solutions to the nonrelativistic Schrödinger equation for the hydrogen atom (and some other systems) is not necessary. The quadratically integrable $l=0$ "anomalous" solution which is usually excluded in this fashion is not a solution to the Schrödinger equation at the origin, and consequently does not need to be excluded by a boundary or other type condition. The relativistic generalization through the Dirac equation of this anomalous solution is not quadratically integrable. The solution generalized according to the relativistic Schrödinger equation is quadratically integrable and obeys the equation at the origin. Consequently, a boundary or other type condition may still be needed in a general relativistic theory.

I. INTRODUCTION

THERE has been recent interest[1–3] in an old problem of quantum mechanics, namely, the fact that a single, uniform condition on the wavefunction appears inadequate to select the proper state functions for the Hamiltonian under consideration.[4,5] If one reviews the discussions of this problem, the situation presented is rather confusing. Kramers[4] states the case for the existence, in the case of the hydrogen atom, of a second solution for $l=0$ (whose series development about the origin behaves like $r^{-(l+1)}$) which does not belong to the hydrogen spectrum but is

* This work supported by the Lockheed Missiles and Space Company Independent Research Program.

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